EXTENSION OF KIRCHHOFF'S THEORY OF ACOUSTIC WAVE PROPAGATION IN A CYLINDERICAL TUBE TO A SLIGHTLY RAREFIED MEDIUM

BY

K. RATHNAM

DEPARTMENT OF MECHANICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

FEBRUARY 1972

EXTENSION OF KIRCHHOFF'S THEORY OF ACOUSTIC WAVE PROPAGATION IN A CYLINDERICAL TUBE TO A SLIGHTLY RAREFIED MEDIUM

A Thesis Submitted
In Partial Fulfilment of the Requirements
for the Degree of
MASTER OF TECHNOLOGY

BY

K. RATHNAM

to the

DEPARTMENT OF MECHANICAL ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY KANPUR

FEBRUARY 1972

CERTIFICATE

This is to certify that the present work has been carried out under my supervision and has not been submitted elsewhere for a degree.

(Madan Mohan Oberai)
Assistant Professor
Department of Mechanical Engineering
Indian Institute of Technology, Kanpur.

ACKNOWLEDGMENTS

I wish to express my sincere thanks to Professor MADAN MOHAN OBERAI for suggesting this interesting problem and for his guidance during the course of this work. I am grateful to Professor Nicholas Rott and Professor M.M.Oberai whose unpublished work (made available by the latter) was of immense help to me. Thanks are also due to Mr. T.M. Narasimhan and Mr. U.N. Sinha for their help and to Mr. C.L. Misra for his neat typing.

ABSTRACT

A theoretical study is made of the Kirchhoff theory of acoustic damping in long cylinderical tubes with the objective of including the corrections due to the velocity slip, the temperature jump and the wall temperature fluctuations. Explicit results have been obtained for the attenuation coefficient and compared with the recent unpublished work of Rott and Oberai. The only assumption made is that the Stokes layer thickness is much smaller than the wave length Λ and this is used latter in the solution of the quadratic equations.

TABLE OF CONTENTS

LIST OF FIGURES AND TABLE

LIST OF SYMBOLS

A_1, A_2, A_3	constants first used in Equation (6)
a	sound velocity, $(\gamma RT_o)^{1/2}$
C_p	specific heat at constant pressure
E_1	constant to be expressed in term of the momentum accommodation coefficient
F_1	constant to be expressed in term of the energy accommodation coefficient
h	complex quantitity first used in equation (6)
i	imaginary quantity, $(-1)^{1/2}$
J_o	Bessel's function of the zero order
K	Wall temperature fluctuation parameter, defined in page 7
m	complex propagation constant
m_1	attenuation coefficient
p	instantaneous acoustic pressure
R	Gas constant
r_w	inner radius of the tube
T	instantaneous acoustic temperature
u	instantaneous axial flow velocity
v	instantaneous radial flow velocity
σ	Prandtl number, $\mu C_p/k$
α	Stokes second viscosity coefficient
μ	dynamic viscosity
ν	kinematic viscosity
ρ	density
ω	frequency of oscillation of the piston
γ	ratio of specific heats, C_p/C_v

λ_g	mean free path of the molecules
$\lambda, \lambda_1, \lambda_2$	constants, defined in page 6
Λ	wave-length, a/ω .

I. INTRODUCTION

The study of small acoustic disturbances into a medium in which the viscous and thermal dissipations are not negligible is essentially a Gasdynamics Problem. Special interest attaches to this problem if the velocity slip and the temperature jump are not ignorable. This case is the subject matter of the present work.

The classical theory of acoustic damping in a long cylinderical tube had its origin over a century ago with the work of Kirchhoff (1868). The above theory that takes into account viscosity at the tube wall is due to Helmholtz (1863). The discrepancy of his theoretical results with the experiments of Kundt (1968) motivate Kirchhoff to present the complete theory that includes the previously neglected effects of heat conduction in the fluid. Starting from the linearised continuity, momentum and energy equations, Kirchhoff obtained an algebraic equation for the propagation constant m. He did this by assuming the solution for all the variables to be of exponential form as regards the axial coordinate and by satisfying the boundary conditions of zero particle velocity and acoustic temperature at the tube wall. The solution of the above algebraic equation gives both the real and the imaginary parts for the propagation constant m. While the real part accounts for the attenuation factor for the waves progressing into the medium, the velocity of propagation of the disturbance can be derived from the imaginary part.

The controversy arising from the disagreement of many experimental results with the classical theory is well known. Decades of scientific research have gone into the study of this problem. The majority of the researchers state that the actual effect of the tube is considerably greater than that given by Kirchhoff's

formula. Starting from the Kirchhoff solution for the acoustic quantities U, V and T, it is possible to get few extra terms for the attenuation coefficient. This is the starting point of the present investigation in which the corrections due to the velocity slip, the temperature jump and the temperature fluctuations at the tube wall are incorporated in the Krichhoff equations.

Among the numerous investigations directed towards finding the causes for the disagreement of the Kirchhoff's results with the experiments, the work of Henry (1931) is notable. He was the first to include the velocity slip, the temperature jump and the temperature fluctuations at the tube wall in the Kirchhoff equations. His calculations were based on physical and mathematical reasonings. He was mainly interested in the study of the ratio of specific heats of gases by the sound velocity method. He incorporated various corrections in his analysis separately one by one and gave explicit formulae for the attenuation coefficient and the velocity of propagation. By neglecting terms of the order $\frac{1}{r_W^2}$ in his calculations, he concludes that the velocity slip has no effect in the attenuation coefficient but adds a small term to the velocity of propagation. In addition, in the determinant expansion, he makes the approximation $m = \frac{h}{a}$ in the terms containing the coefficient $\sqrt{\nu}$. Such approximation affects the terms of the order $\frac{1}{r_W^2}$.

The above investigation was further extended by Weston (1953). He improv the mathematical approximations involved in the classical work and included more terms in the Bessel function expansions. The resistance - concept found in his work for the narrow tubes is worth mentioning. This concept emerges from the work of Lamb (1898). They have included this concept in the analysis for the cas when the tube radius is much smaller than the boundary layer thickness. He also

makes the approximation $m = \frac{h}{a}$ in his analysis [cf. his equation (56)].

The paper by Shields et al. (1965) presents the results of a numerical solution of the Kirchhoff equations for the propagation constant m. They report accurate numerical results for the attenuation coefficient and the velocity of propagation derivable from the constant m. The roots λ_1 and λ_2 are directly obtained by solving numerically the quadratic equation in λ. The first approximate value of m (i.e., $m = \frac{h}{a}$) is incorporated in the argument of the Bessel functions. Then the equation, obtained from the determinant expansion, is solved for a new value of m and is again substituted back into the equation as well as in the argument of the Bessel functions. The iteration is continued until the required accuracy is obtained. They compare the numerical results with some experimental measurements. While the attenuation factor shows good agreement, the velocity of propagation needs corrections due to the velocity slip and the temperature jump, particularly at low pressures.

In an unpublished work, Rott and Oberai obtained new solutions in series form for the Kirchhoff theory of acoustic damping. They employed an expansion scheme with parameter ε, which may be interpreted either as the ratio of the Stokes-layer thickness to the wave length or to the square root of the ratio of the frequency of oscillations to that of collisions of the gas molecules, to solve the problem and incorporated corrections due to the velocity slip, the temperature jump and the wall temperature fluctuations. From the governing equations they have obtained differential equations for the acoustic pressure, the complete solutions of which give the desired results for the attenuation coefficient. The zero- , the first - and the second - order terms are calculated separately. The major simplification of their expansion scheme is that while

the velocity slip and the temperature jump affect the first-order terms, the terms owing to the radial pressure gradient and the axial dissipations do not enter till the second-order of the scheme. In the present work, we show that if the velocity slip, the temperature jump and the wall temperature fluctuations be directly included in the equations originally formulated by Kirchhoff, the results are identical to those of Rott and Oberai.

II. KIRCHHOFF EQUATIONS AND BOUNDARY CONDITIONS

We are concerned with the acoustic oscillation of a perfect gas contained in a semi-infinite cylinderical tube. The gas is driven by the periodic motion of a piston located at the origin of the co-ordinate system being employed. The aim of the present investigation is to obtain the attenuation coefficient of this small disturbance. In the actual case, one needs in the governing equations both the dissipative and the non-linear convective terms. Since the undisturbed gas is at rest, the velocity components are directly given by their acoustic quantities which are considered to be infinitesimally small; consequently, all the non-linear terms drop out.

In the cylinderical co-ordinate system (x,r,θ), the axi-symmetric time-dependent equation of continuity, the Navier-Stokes equations for the momentum and energy are :

$$\frac{1}{\rho_o}\frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_o}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right] + \alpha\nu\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r}\right] \tag{2}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_o}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right] + \alpha\nu\frac{\partial}{\partial r}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r}\right] \tag{3}$$

$$\frac{\partial T}{\partial t} = \frac{1}{\rho_o C_p}\frac{\partial p}{\partial t} + \frac{\nu}{\sigma}\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right] \tag{4}$$

Associated with the above four equations is the equation of state of the perfect (thermally as well as calorically) gas :

$$p = R \rho T \tag{5}$$

[of course, the assumption of a perfect gas is not essential]

Following Kirchhoff, the expressions for the velocity components u and v and the temperature T, which are assumed to behave like a function of r multiplied by e^{mx+ht}, are given as

$$u = \{A_1 J_o [r(m^2 - \frac{h}{\nu})^{\frac{1}{2}}] + A_2 m [\frac{\gamma\nu}{\sigma} - \frac{h}{\lambda_1}] J_o [r(m^2 - \lambda_1)^{\frac{1}{2}}]$$

$$+ A_3 m [\frac{\gamma\nu}{\sigma} - \frac{h}{\lambda_2}] J_o [r(m^2 - \lambda_2)^{\frac{1}{2}}]\} e^{mx+ht} \tag{6}$$

$$v = \{\frac{m}{(m^2 - \frac{h}{\nu})} A_1 \frac{\partial}{\partial r} J_o [r(m^2 - \frac{h}{\nu})^{\frac{1}{2}}] + A_2 [\frac{\gamma\nu}{\sigma} - \frac{h}{\lambda_1}] \frac{\partial}{\partial r} J_o [r(m^2 - \lambda_1)^{\frac{1}{2}}]$$

$$+ A_3 [\frac{\gamma\nu}{\sigma} - \frac{h}{\lambda_2}] \frac{\partial}{\partial r} J_o [r(m^2 - \lambda_2)^{\frac{1}{2}}]\} e^{mx+ht} \tag{7}$$

$$T = T_o (\gamma-1) \{A_2 J_o[r(m^2 - \lambda_1)^{\frac{1}{2}}] + A_3 J_o [r(m^2 - \lambda_2)^{\frac{1}{2}}]\} e^{mx+ht} \tag{8}$$

where λ_1 and λ_2 are the small and large roots of the equation

$$h^2 - [a^2 + \nu h + \nu h \alpha + \frac{\nu h \gamma}{\sigma}] \lambda + \frac{\gamma\nu}{\sigma h} [\frac{a^2}{\gamma} + \nu h + \nu h \alpha] \lambda^2 = 0 \tag{9}$$

Kirchhoff obtained the above solution using the only boundary condition that along the axis of the tube, the various quantities remain finite. The detailed derivation of these equations is given in the book Theory of Sound Vol. II by Lord Rayleigh.

Boundary conditions :

The propagation constant m was determined by Kirchhoff by using the boundary conditions of zero particle velocity and acoustic temperature at the tube wall. However, as stated in the introduction, the motivation of the present work is to incorporate the corrections due to the velocity slip, the temperature jump and the temperature fluctuations at the tube wall in the Kirchhoff equations. The wall temperature fluctuations are governed by a parameter K, defined by $\left(\frac{k\rho C_p}{k_s\rho_s C_s}\right)^{1/2}$, which becomes significant at low temperature due to T_s^3 dependence of C_s (k_s, ρ_s and C_s are respectively the thermal conductivity, the density and the specific heat of the tube material). While, the phenomena of slip and temperature jump are the part of rarefied gasdynamics corresponding to slight departure from the continuum limit. These effects which appear as small corrections to the continuum no slip, no temperature jump boundary conditions are significant when the pressure and consequently density of the gas is low. The corresponding expressions for these corrections are given below :

i) Wall temperature fluctuations

We need to solve the heat conduction equation in the tube wall. Let $T_s\, e^{mx+ht}$ be the fluctuating temperature at a point in the tube material at a distance y from the inner surface of the tube wall. See Fig. 1. Two assumptions are made here, namely the axial heat conduction is negligible in comparison to the radial heat conduction and the inner surface is considered as plane instead of cylinderical; the small penetration of the heat flux from the gas into the wall provides justification for these assumptions (cf. Ref. 2)

We then have

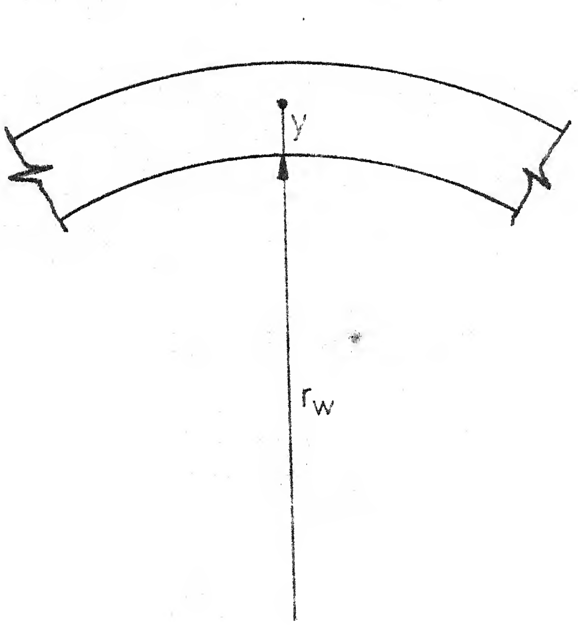

FIG. 1. TUBE WALL

$$\frac{\partial^2 T_s}{\partial y^2} = \frac{h\ \rho_s\ C_s}{k_s} T_s \tag{10}$$

with the boundary conditions $T_s = 0$, $y \to \infty$ and $T_s = T_W$, $y = 0$ (11)

where $T_W\ e^{mx+ht}$ is the instantaneous wall temperature.

The solution of the equation (10) may be obtained as

$$T_s = T_W\ e^{-\left(\frac{\rho_s\ C_s\ h}{k_s}\right)^{1/2} y} \tag{12}$$

Making a heat balance along the inner surface of the tube wall, we have

$$- k \left.\frac{\partial T}{\partial r}\right|_{r=r_W} = - k_s \left.\frac{\partial T_s}{\partial y}\right|_{y=0} \tag{13}$$

After simplification, we obtain

$$T_W = - K \left(\frac{\nu}{\sigma h}\right)^{\frac{1}{2}} \left.\frac{\partial T}{\partial r}\right|_{r=r_W} \tag{14}$$

ii) Temperature jump

The statement of the temperature jump at the tube wall is as follows

$$T - T_W = - F_1\ \lambda_g \left.\frac{\partial T}{\partial r}\right|_{r=r_W} \tag{15}$$

On substituting for T_W from the equation (14), the following boundary condition is obtained :

$$(T)_{r=r_w} = - [K (\frac{\nu}{\sigma h})^{\frac{1}{2}} + F_1 \lambda_g] \frac{\partial T}{\partial r}\Big|_{r=r_w} \tag{16}$$

iii) Slip

The prescribed boundary conditions at the surface of the wall are that the normal velocity is zero; however, due to slip the tangential velocity is proportional to the gradient $\frac{\partial u}{\partial r}$. Explicitly,

$$v = 0 \tag{17}$$

$$(u)_{r=r_w} = - E_1 \lambda_g \frac{\partial u}{\partial r}\Big|_{r=r_w} \tag{18}$$

III. METHOD OF SOLUTION

The qualitative as well as quantitative effects of the factors described in the previous chapter can be obtained by using the modified boundary conditions derived in that chapter. The corrections are sought in the various acoustic quantities u, v and T. The corresponding expressions after satisfying the boundary conditions at the tube wall will give

$$A_1\{J_o[r(m^2 - \frac{h}{\nu})^{\frac{1}{2}}] + E_1 \lambda_g \frac{\partial}{\partial r} J_o[r(m^2 - \frac{h}{\nu})^{\frac{1}{2}}]\} + A_2\, m\, (\frac{\gamma\nu}{\sigma} - \frac{h}{\lambda_1})\{J_o[r(m^2 - \lambda_1)^{\frac{1}{2}}]$$

$$+ E_1 \lambda_g \frac{\partial}{\partial r} J_o\, [r(m^2 - \lambda_1)^{\frac{1}{2}}]\} + A_3\, m\, (\frac{\gamma\nu}{\sigma} - \frac{h}{\lambda_2})\{J_o[r(m^2 - \lambda_2)^{\frac{1}{2}}]$$

$$+ E_1 \lambda_g \frac{\partial}{\partial r} J_o\, [r(m^2 - \lambda_2)^{\frac{1}{2}}]\} = 0 \tag{19}$$

$$A_1 \frac{m}{(m^2 - \frac{h}{\nu})} \frac{\partial}{\partial r} J_o[r(m^2 - \frac{h}{\nu})^{\frac{1}{2}}] + A_2\, (\frac{\gamma\nu}{\sigma} - \frac{h}{\lambda_1}) \frac{\partial}{\partial r} J_o[r(m^2 - \lambda_1)^{\frac{1}{2}}]$$

$$+ A_3 (\frac{\gamma\nu}{\sigma} - \frac{h}{\lambda_2}) \frac{\partial}{\partial r} J_o[r(m^2 - \lambda_2)^{\frac{1}{2}}] = 0 \tag{20}$$

$$A_2\, \{J_o[r(m^2 - \lambda_1)^{\frac{1}{2}}] + [F_1 \lambda_g + K(\frac{\nu}{\sigma h})^{\frac{1}{2}}] \frac{\partial}{\partial r} J_o[r(m^2 - \lambda_1)^{\frac{1}{2}}]\}$$

$$+ A_3\, \{J_o[r(m^2 - \lambda_2)^{\frac{1}{2}}] + [F_1 \lambda_g + K(\frac{\nu}{\sigma h})^{\frac{1}{2}}] \frac{\partial}{\partial r} J_o[r(m^2 - \lambda_2)^{\frac{1}{2}}]\} = 0 \tag{21}$$

Writing them in the matrix form with A_1, A_2, and A_3 as column vectors, we get $[M]\ [A] = [0]$. (22)

The coefficient matrix M contains the unknowns λ_1, λ_2 and the parameter m. The method of approach for obtaining the propagation constant m is similar to

The coefficient matrix M

$$\begin{bmatrix} Q\,\{1 + \frac{E\nu}{a}\frac{\partial}{\partial r}\ln Q\} & Q_1\, m\,(\frac{\gamma\nu}{\sigma} - \frac{h}{\lambda_1})\,\{1 + \frac{E\nu}{a}\frac{\partial}{\partial r}\ln Q_1\} & Q_2\, m(\frac{\gamma\nu}{\sigma} - \frac{h}{\lambda_2})\,\{1 + \frac{E\nu}{a}\frac{\partial}{\partial r}\ln Q_2\} \\ Q\,\frac{m}{(m^2 - \frac{h}{\nu})}\,\frac{\partial}{\partial r}\ln Q & Q_1(\frac{\gamma\nu}{\sigma} - \frac{h}{\lambda_1})\,\frac{\partial}{\partial r}\ln Q_1 & Q_2\,(\frac{\gamma\nu}{\sigma} - \frac{h}{\lambda_2})\,\frac{\partial}{\partial r}\ln Q_2 \\ 0 & Q_1\,\{1 + [\frac{F\nu}{a} + K(\frac{\nu}{\sigma h})^2]^{\frac{1}{2}}\,\frac{\partial}{\partial r}\ln Q_1\} & Q_2\,\{1 + [\frac{F\nu}{a} + K(\frac{\nu}{\sigma h})^2]^{\frac{1}{2}}\,\frac{\partial}{\partial r}\ln Q_2\} \end{bmatrix}$$

where $Q = J_o[r(m^2 - \frac{h}{\nu})^{\frac{1}{2}}]$, $Q_1 = J_o\,[r(m^2 - \lambda_1)^{\frac{1}{2}}]$, $Q_2 = J_o\,[r(m^2 - \lambda_2)^{\frac{1}{2}}]$,

$E = E_1\,(\frac{\pi\gamma}{2})^{\frac{1}{2}}$ and $F = F_1\,(\frac{\pi\gamma}{2})^{\frac{1}{2}}$.

that of Kirchhoff. The condition that the determinant of the coefficient matrix should vanish for the non-zero value of the vector $[A_1, A_2, A_3]$ gives an algebraic equation for the propagation constant. In interpreting the solution there are two special cases of interest. The first, wide tube case is the same as that considered by Kirchhoff. The explanations given by Kirchhoff showed that the approximate values of λ_1 and m^2 are both equal to $\frac{h^2}{a^2}$. From this he concludes that the term '$r(m^2 - \lambda_1)^{1/2}$' appearing as an argument in one of the Bessel functions is to be regarded as small. However, the other two arguments '$r(m^2 - \frac{h}{\nu})^{1/2}$' and '$r(m^2 - \lambda_2)^{1/2}$' are large for this case. In the other extreme case of narrow tube, all the three arguments are to be regarded as small. The relevant asymptotic expansions of the Bessel function are given below :

a) When $|Z|$ is large and such that its imaginary part is positive

$$\frac{\partial}{\partial Z} \ln J_o(Z) = -i \; [1 + \frac{1}{8Z^2} + \ldots] - [\frac{1}{2Z} - \frac{1}{8Z^3} \ldots] \qquad (23)$$

b) When $|Z|$ is small

$$\frac{\partial}{\partial Z} \ln J_o(Z) = -\frac{Z}{2} \; [1 + \frac{Z^2}{8} + \frac{Z^4}{48} + \frac{11 \; Z^6}{3042} + \ldots] \qquad (24)$$

Having discussed the approximations involved in the series expansions of the zero order Bessel function, the next task is to obtain the roots λ_1 and λ_2. These roots, appearing in the Kirchhoff equations, are independent of the tube radius, being determined by the equation

$$B \; \lambda^2 - C \; \lambda + D = 0 \qquad (25)$$

$$\left.\begin{aligned} \text{where} \quad B &= \frac{\gamma\nu}{\sigma h}\left\{\frac{a^2}{\gamma} + \nu h\ (1+\alpha)\right\} \\ C &= \left\{a^2 + \nu h\ \left(1 + \alpha + \frac{\gamma}{\sigma}\right)\right\} \\ D &= h^2 \end{aligned}\right] \tag{26}$$

Solving the quadratic equation (25), we get

$$\lambda_{1,2} = \frac{C}{2B} \pm \frac{C}{2B}\left[1 - \frac{2BD}{C^2} - \frac{2B^2D^2}{C^4} + \text{higher order terms}\right] \tag{27}$$

In the binomial expansion for the expression under the square root sign, we have used the fact, $\left|\frac{BD}{C^2}\right| \ll 1$. (From the form of the expressions for B, C and D, one may show that this condition is fulfilled if the Stokes layer thickness is much smaller than the wave length).

Substituting for B, C, and D in the equation (27) and after simplification, we obtain

$$\lambda_1 = \frac{h^2}{a^2}\left\{1 - \frac{\nu h}{a^2}\left(1 + \alpha + \frac{\gamma-1}{\sigma}\right)\right\} \tag{28}$$

$$\lambda_2 = \frac{\sigma h}{\nu}\left\{1 - \frac{\nu h}{a^2}(\gamma-1)\left(1 + \alpha - \frac{1}{\sigma}\right)\right\} \tag{29}$$

i) <u>WIDE TUBE</u>

In the previous section we have mentioned the Kirchhoff approximations in the Bessel function expansion with regard to its argument. Our aim now would be to include these approximations in the Kirchhoff determinant (determinant of the coefficient matrix M) before expanding it. For the present case, we need the asymptotic expansions of the zero order Bessel function given by

equations (23) and (24). In the expansions given below we have retained two terms instead of one as given in Rayleigh's book. A comparison of the results of the present investigation with the recent unpublished work of Rott and Oberai is not possible without the second term appearing in all the three expansions. Hence we have

$$\left.\begin{aligned}
&\frac{\partial}{\partial r} \ln Q = \left(\frac{h}{\nu}\right)^{\frac{1}{2}} - \frac{1}{2r} \\
&\frac{\partial}{\partial r} \ln Q_1 = - \frac{r}{2} (m^2 - \lambda_1) - \frac{r^3}{16} (m^2 - \lambda_1)^2 \\
&\frac{\partial}{\partial r} \ln Q_2 = (\lambda_2)^{\frac{1}{2}} - \frac{1}{2r}
\end{aligned}\right] \qquad (30)$$

Incorporating these expressions in the coefficient matrix M, we expand its determinant. This, after division by $Q\,Q_1\,Q_2$ gives

$$\left[1 + \frac{E\nu}{a}\left[\left(\frac{h}{\nu}\right)^{\frac{1}{2}} - \frac{1}{2r_W}\right]\right]\left[\left[\frac{\gamma\nu}{\sigma} - \frac{h}{\lambda_1}\right]\left[- \frac{r_W}{2}(m^2 - \lambda_1) - \frac{r_W^3}{16}(m^2 - \lambda_1)^2\right]\right.$$

$$\left[1 + \left\{\frac{F\nu}{a} + K\left(\frac{\nu}{\sigma h}\right)^{\frac{1}{2}}\right\}\left\{(\lambda_2)^{\frac{1}{2}} - \frac{1}{2r_W}\right\}\right] - \left[\frac{\gamma\nu}{\sigma} - \frac{h}{\lambda_2}\right]\left[(\lambda_2)^{\frac{1}{2}} - \frac{1}{2r_W}\right]\left[1 + \left\{\frac{F\nu}{a} + K\left(\frac{\nu}{\sigma h}\right)^{\frac{1}{2}}\right\}\right.$$

$$\left.\left.\left\{- \frac{r_W}{2}(m^2 - \lambda_1)\right\}\right]\right] - \frac{m^2}{(m^2 - \frac{h}{\nu})}\left[\left(\frac{h}{\nu}\right)^{\frac{1}{2}} - \frac{1}{2r_W}\right]\left[\frac{\gamma\nu}{\sigma} - \frac{h}{\lambda_1}\right]\left[1 - \frac{E\nu}{a}\frac{r_W}{2}(m^2 - \lambda_1)\right]$$

$$\left[1 + \left\{\frac{F\nu}{a} + K\left(\frac{\nu}{\sigma h}\right)^{\frac{1}{2}}\right\}\left\{(\lambda_2)^{\frac{1}{2}} - \frac{1}{2r_W}\right\}\right] + \frac{m^2}{(m^2 - \frac{h}{\nu})}\left[\left(\frac{h}{\nu}\right)^{\frac{1}{2}} - \frac{1}{2r_W}\right]\left[\frac{\gamma\nu}{\sigma} - \frac{h}{\lambda_2}\right]$$

$$\left[1 + \frac{E\nu}{a}\left\{(\lambda_2)^{\frac{1}{2}} - \frac{1}{2r_W}\right\}\right]\left[1 + \left\{\frac{F\nu}{a} + K\left(\frac{\nu}{\sigma h}\right)^{\frac{1}{2}}\right\}\left\{- \frac{r_W}{2}(m^2 - \lambda_1)\right\}\right] = 0 \qquad (31)$$

By using the approximations (28) & (29) for λ_1 & λ_2 and simplifying we obtain

$$[1 + \frac{E\nu}{a}\{(\frac{h}{\nu})^{\frac{1}{2}} - \frac{1}{2r_W}\}]\ [\frac{r_W\ a^2}{2\ \ h}\{(m^2 - \lambda_1) + \frac{r_W^2}{8}(m^2 - \lambda_1)^2\}\ \{1 + \frac{F\nu}{a(1+K)}(\frac{\sigma h}{\nu})^{\frac{1}{2}}$$

$$- \frac{F\nu}{a(1+K)}\frac{1}{2r_W} - \frac{K}{(1+K)}(\frac{\nu}{\sigma h})^{\frac{1}{2}}\frac{1}{2r_W}\} - \frac{(\gamma-1)\nu}{\sigma(1+K)}\{(\frac{\sigma h}{\nu})^{\frac{1}{2}} - \frac{1}{2r_W}\}\ \{1 - \frac{r_W}{2}K(\frac{\nu}{\sigma h})^{\frac{1}{2}}(m^2 - \lambda_1)\}]$$

$$- m^2\frac{\nu a^2}{h^2}[(\frac{h}{\nu})^{\frac{1}{2}} - \frac{1}{2r_W}]\ [1 + \frac{F\nu}{a(1+K)}(\frac{\sigma h}{\nu})^{\frac{1}{2}} - \frac{F\nu}{a(1+K)}\frac{1}{2r_W} - \frac{K}{(1+K)}(\frac{\nu}{\sigma h})^{\frac{1}{2}}\frac{1}{2r_W}] = 0$$

... (32)

The next task would be to obtain the quartic equation and then, its solution. Those are as follows :

$$\frac{r_W^2}{8}m^4 + [1 - \frac{r_W^2\ h^2}{4\ a^2}\{1 - \frac{\nu h}{a^2}(1 + \alpha + \frac{\gamma-1}{\sigma})\} - \frac{2}{r_W}(\frac{\nu}{h})^{\frac{1}{2}}\{1 - \frac{1}{2r_W}(\frac{\nu}{h})^{\frac{1}{2}}\}\ \{1 - \frac{E(\nu h)^{\frac{1}{2}}}{a}$$

$$+ \frac{E\nu}{2a\ r_W} + \frac{E^2\ \nu h}{a^2}\} + \frac{\nu h\ K(\gamma-1)}{a^2\ \sigma(1+K)}]\ m^2 - \frac{h^2}{a^2}[1 - \frac{\nu h}{a^2}(1 + \alpha + \frac{\gamma-1}{\sigma}) - \frac{r_W^2\ h^2}{8\ a^2}$$

$$\{1 - \frac{2\nu h}{a^2}(1 + \alpha + \frac{\gamma-1}{\sigma})\} + \frac{\nu h\ K(\gamma-1)}{a^2\ \sigma(1+K)} + \frac{2}{r_W}(\frac{\nu}{h})^{\frac{1}{2}}\ \frac{(\gamma-1)}{\sigma(1+K)}\{\sqrt{\sigma} - \frac{1}{2r_W}(\frac{\nu}{h})^{\frac{1}{2}}\}$$

$$\{1 - \frac{F}{a(1+K)}(\sigma\nu h)^{\frac{1}{2}} + \frac{F\nu}{2a\ r_W(1+K)} + \frac{K}{2r_W(1+K)}(\frac{\nu}{\sigma h})^{\frac{1}{2}} - \frac{F\ \nu K}{a r_W(1+K)^2} + \frac{F^2\ \nu h\sigma}{a^2(1+K)^2}\}] = 0$$

... (33)

Hence

$$m^2 = \frac{h^2}{a^2}\left[1 - \frac{\nu h}{a^2}\left\{\frac{3}{2} + \alpha + \frac{\gamma-1}{\sigma} + \frac{(\gamma-1)^2}{2\sigma(1+K)^2} + \frac{(\gamma-1)}{(\sigma)^{\frac{1}{2}}(1+K)}\right\} + \frac{2}{r_W}\left(\frac{\nu}{h}\right)^{\frac{1}{2}}\left\{1 + \frac{(\gamma-1)}{(\sigma)^{\frac{1}{2}}(1+K)}\right\}\right.$$

$$+ \frac{\nu}{r_W^2 h}\left\{3 + \frac{4(\gamma-1)}{(\sigma)^{\frac{1}{2}}(1+K)} - \frac{(\gamma-1)}{\sigma(1+K)^2}\right\} - \frac{2\ E\nu}{a\ r_W}\left\{1 + \frac{3}{r_W}\left(\frac{\nu}{h}\right)^{\frac{1}{2}} + \frac{2}{r_W}\left(\frac{\nu}{h\sigma}\right)^{\frac{1}{2}}\frac{(\gamma-1)}{(1+K)}\right\}$$

$$- \frac{2\ F(\gamma-1)\nu}{a(1+K)^2\ r_W}\left\{1 + \frac{2}{r_W}\left(\frac{\nu}{h}\right)^{\frac{1}{2}} - \frac{1}{r_W}\left(\frac{\nu}{h\sigma}\right)^{\frac{1}{2}}\frac{1}{(1+K)}\right\} + \ldots + \frac{2}{r_W}\frac{\nu(\nu h)^{\frac{1}{2}}}{a^2}$$

$$\left.\left\{E^2 + \frac{F^2(\sigma)^{\frac{1}{2}}(\gamma-1)}{(1+K)^3}\right\}\right] = 0 \tag{34}$$

Out of the two possible roots given by equation (33), we have chosen the one which becomes identical with the value of m^2 obtained by neglecting viscosity and thermal conductivity in the governing equations. Putting $h = i\omega$ (so that $\left(\frac{1}{h}\right)^{\frac{1}{2}} = \left(\frac{1}{2\omega}\right)^{\frac{1}{2}}(1-i)$ in the equation (34), we get

$$m = \pm\frac{i\omega}{a}\left[1 + \frac{(1-i)}{r_W}\left(\frac{\nu}{2\omega}\right)^{\frac{1}{2}}\left\{1 + \frac{(\gamma-1)}{(\sigma)^{\frac{1}{2}}(1+K)}\right\} + \frac{\nu}{r_W^2\ i\omega}\left\{1 + \frac{(\gamma-1)}{(\sigma)^{\frac{1}{2}}(1+K)} - \frac{\gamma(\gamma-1)}{2\sigma(1+K)^2}\right\}\right.$$

$$- \frac{E\nu}{a\ r_W}\left\{1 + \frac{(1-i)}{r_W}\left(\frac{\nu}{2\omega}\right)^{\frac{1}{2}}\left(2 + \frac{(\gamma-1)}{(\sigma)^{\frac{1}{2}}(1+K)}\right)\right\} - \frac{F\nu\ (\gamma-1)}{a\ r_W(1+K)^2}\left\{1 - \frac{(1-i)}{r_W}\left(\frac{\nu}{2\omega}\right)^{\frac{1}{2}}\right.$$

$$\left.\left(\frac{\gamma}{(\sigma)^{\frac{1}{2}}(1+K)} - 1\right)\right\} - \frac{i\omega\nu}{a^2}\left\{\frac{3}{4} + \frac{\alpha}{2} + \frac{\gamma-1}{2\sigma} + \frac{(\gamma-1)^2}{4\sigma(1+K)^2} + \frac{(\gamma-1)}{2(\sigma)^{\frac{1}{2}}(1+K)}\right\} + \ldots\ldots$$

$$\left.\ldots + \frac{(1+i)\nu}{r_W^2\ a^2}\left(\frac{\nu\omega}{2}\right)^{\frac{1}{2}}\left\{E^2 + \frac{F^2(\gamma-1)(\sigma)^{\frac{1}{2}}}{(1+K)^3}\right\}\right] \tag{35}$$

Separating the real part from the above complex propagation constant and choosing the negative sign (corresponds to damping), we have the following expression for the attenuation coefficient

$$m_1 = -\frac{\omega}{a}\Bigg[\frac{(2)^{\frac{1}{2}} C_o}{|\eta_W|} + \frac{2C_o - \frac{\gamma(\gamma-1)}{2\sigma(1+K)^2}}{|\eta_W^2|} + \left|\frac{\varepsilon}{\eta_W^2}\right| \frac{1}{(2)^{\frac{1}{2}}}[-E(1+2C_o)$$

$$+ \frac{F(\gamma-1)}{(1+K)^2}\{\frac{\gamma}{(\sigma)^{\frac{1}{2}}(1+K)} - 1\}] + \frac{|\varepsilon^2|}{2}\{\frac{3}{2} + \alpha + \frac{(\gamma-1)}{\sigma} + \frac{(\gamma-1)^2}{2\sigma(1+K)^2} + \frac{(\gamma-1)}{(\sigma)^{\frac{1}{2}}(1+K)}\}$$

$$+ \dots - \left|\frac{\varepsilon^2}{\eta_W}\right| \frac{1}{(2)^{\frac{1}{2}}}\{E^2 + \frac{F^2(\gamma-1)(\sigma)^{\frac{1}{2}}}{(1+K)^3}\}\Bigg] \tag{36}$$

where $C_o = \frac{1}{2}[1 + \frac{(\gamma-1)}{(\sigma)^{\frac{1}{2}}(1+K)}]$

$$|\eta_W| = r_W \left(\frac{\omega}{\nu}\right)^{\frac{1}{2}} \quad \text{and} \quad |\varepsilon| = \left(\frac{\omega\nu}{a^2}\right)^{\frac{1}{2}}$$

ii) NARROW TUBE

This case arises when the layer affected by friction, instead of forming a thin coating to the walls, extends itself over the whole section. This happens when the Stokes-layer thickness, defined by $(\frac{\nu}{\omega})^{1/2}$, is comparable to the tube radius. In view of the assumption $r_W \sim (\frac{\nu}{\omega})^{1/2}$ the argument of all Bessel functions in the matrix M, becomes small. Hence, a fresh start is necessary for the present case. It may be pointed out that the method discussed

by Rott and Oberai leads to results valid for arbitrary ratios of Stokes-layer thickness to the tube radius. In the expressions given below, we have to retain three terms of the expansions in order to compare present results with those of Rott and Oberai. Hence, we have

$$\left.\begin{aligned}
\frac{\partial}{\partial r}\ln Q &= -\frac{r}{2}\left(m^2-\frac{h}{\nu}\right)\left[1+\frac{r^2}{8}\left(m^2-\frac{h}{\nu}\right)+\frac{r^4}{48}\left(m^2-\frac{h}{\nu}\right)^2+\,..\,\right]\\
\frac{\partial}{\partial r}\ln Q_1 &= -\frac{r}{2}\left(m^2-\lambda_1\right)\left[1+\frac{r^2}{8}\left(m^2-\lambda_1\right)+\frac{r^4}{48}\left(m^2-\lambda_1\right)^2+\,..\,\right]\\
\frac{\partial}{\partial r}\ln Q_2 &= -\frac{r}{2}\left(m^2-\lambda_2\right)\left[1+\frac{r^2}{8}\left(m^2-\lambda_2\right)+\frac{r^4}{48}\left(m^2-\lambda_2\right)^2+\,..\,\right]
\end{aligned}\right]\qquad(37)$$

After incorporating these expressions in the coefficient matrix M, we expand its determinant. This, after simplification and division by $Q\ Q_1\ Q_2$ gives

$$\begin{aligned}
&(m^2-\lambda_1)+\frac{r_W^2}{8}(m^2-\lambda_1)^2+\frac{r_W^4}{48}(m^2-\lambda_1)^3+\frac{(\gamma-1)}{\sigma}\frac{\nu h}{a^2}\left[1-\frac{2\nu h}{a^2}\left(1+\alpha-\frac{1}{\sigma}\right)\right]\\
&\left[(m^2-\lambda_2)+\frac{r_W^2}{8}(m^2-\lambda_2)^2+\frac{r_W^4}{48}(m^2-\lambda_2)^3\right]\left[1-\frac{F\nu\, r_W\lambda_2}{2a}+\frac{F^2\nu^2 r_W^2\lambda_2^2}{4a^2}\right]\\
&-m^2\left[1+\frac{r_W^2}{8}m^2-\frac{r_W^2 h}{8\nu}+\frac{r_W^4}{48}m^4-\frac{r_W^4 h}{24\nu}m^2+\frac{r_W^2 h^2}{48\nu^2}\right]\left[\left\{1-\frac{E\, r_W h}{2a}+\frac{E^2 r_W^2 h^2}{4a^2}\right\}\right.\\
&\left.+\frac{(\gamma-1)\nu h}{\sigma\ a^2}\left\{1-2\frac{\nu h}{a^2}\left(1+\alpha-\frac{1}{\sigma}\right)\right\}\left\{1-\frac{F\nu\, r_W\lambda_2}{2a}+\frac{F^2\nu^2 r_W^2\lambda_2^2}{4a^2}\right\}\right]=0
\end{aligned}\qquad(38)$$

On further simplification, we obtain

$$[\frac{r_W^4\ h}{24\ \nu}\{1+\frac{(\gamma-1)}{\sigma}\frac{\nu h}{a^2}\}+\frac{r_W^2}{8}(\frac{E\ r_W\ h}{2a})]\ m^4+[\frac{r_W^2\ h}{8\nu}+\frac{E\ r_W\ h}{2a}+\frac{r_W^2\ h(\gamma-1)}{8\nu}\frac{\nu h}{\sigma}\frac{\nu h}{a^2}$$

$$-\frac{r_W^2}{4}\lambda_1-\frac{r_W^2}{4}\lambda_2\frac{(\gamma-1)}{\sigma}\frac{\nu h}{a^2}]\ m^2-[\lambda_1+\lambda_2\frac{(\gamma-1)}{\sigma}\frac{\nu h}{a^2}\{1-\frac{2\nu h}{a^2}(1+\alpha-\frac{1}{\sigma})\}$$

$$\{1-\frac{F\nu\ r_W\ \lambda_2}{2a}+\frac{F^2\ \nu^2\ r_W^2\ \lambda_2^2}{4a^2}\}]=0 \tag{39}$$

Solving this quartic equation and choosing the proper root, we get

$$m^2=\frac{8\ \nu h\ \gamma}{r_W^2\ \ a^2}[1-\frac{4E\ \nu}{r_W\ a}+\frac{16E^2\nu^2}{r_W^2\ \ a^2}-\frac{F\ r_W\ \sigma h(\gamma-1)}{2a\qquad\gamma}+\frac{F^2\ r_W^2\ \sigma^2\ h^2(\gamma-1)}{4a^2\qquad\gamma}$$

$$-\frac{\nu h\gamma}{a^2}(\frac{5}{3}+\alpha)] \tag{40}$$

by using the approximations (28) & (29) for λ_1 & λ_2. Simplifying, we obtain

$$m=\pm\frac{2(2\nu h\gamma)^{\frac{1}{2}}}{r_W\ a}[1-\frac{2E\ \nu}{r_W\ a}+\frac{6E^2\ \nu^2}{r_W^2\ \ a^2}-\frac{F\ r_W\ \sigma(\gamma-1)h}{4a\qquad\gamma}+\frac{F^2\ r_W^2\ \sigma^2\ h^2(\gamma-1)}{8a^2\qquad\gamma}$$

$$-\frac{\nu h\gamma}{a^2\ 2}(\frac{5}{3}+\alpha)] \tag{41}$$

Putting $h = i\omega$ in the above equation, we get

$$m=\pm\frac{2(1+i)\ (\nu\omega\gamma)^{\frac{1}{2}}}{a\ r_W}[1-\frac{2E\ \nu}{r_W\ a}+\frac{6E^2\ \nu^2}{r_W^2\ \ a^2}-\frac{F\ r_W\ \sigma(\gamma-1)\ i\omega}{4a\qquad\gamma}-\frac{F^2\ r_W^2\ \sigma^2(\gamma-1)\omega^2}{8\ a^2\qquad\gamma}$$

$$-\frac{i\omega\nu\gamma}{2a^2}(\frac{5}{3}+\alpha)] \tag{42}$$

From this, the expression for the attenuation coefficient, which corresponds to damping, is obtained

$$m_1 = - \frac{\omega}{a} \left[\frac{2(\gamma)^{\frac{1}{2}}}{|n_W|} - 4 \left|\frac{\varepsilon}{n_W^2}\right| E\,(\gamma)^{\frac{1}{2}} + 12 \left|\frac{\varepsilon^2}{n_W^3}\right| E^2(\gamma)^{\frac{1}{2}} + |\varepsilon| \frac{F\sigma(\gamma-1)}{2\,(\gamma)^{1/2}} \right.$$

$$\left. - |\varepsilon^2\, n_W| \frac{F^2\, \sigma^2(\gamma-1)}{4(\gamma)^{1/2}} + \left|\frac{\varepsilon^2}{n_W}\right| \left(\frac{5}{3} + \alpha\right) \gamma(\gamma)^{\frac{1}{2}} \right] \tag{43}$$

where $|n_W| = r_W \left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}$

$$|\varepsilon| = \left(\frac{\omega\nu}{a^2}\right)^{\frac{1}{2}} .$$

IV DISCUSSION

We shall summarise the important results obtained in the previous chapter and give comparison, whenever possible, with those obtained by other authors.

i) WIDE TUBE

The effect of the wall temperature fluctuation parameter K is to decrease the thermal contribution to damping. The slip has a negative contribution to damping. The sign of the contribution of the temperature jump depends on magnitude of K relative to $(\frac{\gamma}{\sqrt{\sigma}} - 1)$. See Table 1. Henry and Weston have studied the effect of slip and temperature jump and concluded that they have no effect on damping. In fact, these authors did not carry out the Bessel function expansions to the order $\frac{1}{r_W^2}$ (as also pointed out by Rott and Oberai); this has been done in the present work. However, the next higher order terms with the coefficient $|\varepsilon^2|$ agree with those of Weston. In Fig. 2 we compare our results with the experimental measurements reported in ref . 4 for Argon.

ii) NARROW TUBE

The slip has a large and negative contribution to damping. This result agrees with that of Weston. The contribution of the temperature jump is positive but of much smaller order.

Though the present method involves more mathematical calculations than that of Henry and Weston, the generality that no approximation of the type assumed by these authors has been used justifies it. As stated earlier, present results are identical with those of Rott and Oberai. Finally, it may be pointed out that the contribution of the term ' $\frac{\nu h}{a^2}(\gamma-1)(1+\alpha-\frac{1}{\sigma})$ ' appearing in the root λ_2 starts with the coefficient $|\frac{\varepsilon^2}{\eta_W}|$ in both the cases and hence cannot be neglected in the analysis.

VALUES OF K FOR HELIUM

Temp °K	Helium C_p	Helium k	Helium ρ	Glass C_s	Glass k_s	Glass ρ_s	Copper C_s	Copper k_s	Copper ρ_s	K	
	J/gm°K	Watt/cm°K	gm/cm^3	J/gm°K	Watt/cm°K	gm/cm^3	J/gm°K	Watt/cm°K	gm/cm^3	Glass	Copper
4	3.99	0.000262	0.1290	0.000201	0.00100	3.0	0.000091	3.1000	8.96	14.960	0.2310
6	5.196	0.000110	0.0081	0.000753	0.00110	3.0	0.00023	4.7000	8.96	1.366	0.0219
8	5.196	0.000135	0.0061	0.00209	0.00115	3.0	0.00047	6.1000	8.96	0.7703	0.0129
10	5.196	0.000150	0.00488	0.00419	0.00120	3.0	0.00086	7.9000	8.96	0.5021	0.0079
15	5.196	0.000200	0.00325	0.0137	0.00130	3.0	0.00270	11.0000	8.96	0.2514	0.0036
20	5.196	0.000250	0.00244	0.0274	0.00150	3.0	0.00770	13.000	8.96	0.1603	0.0019

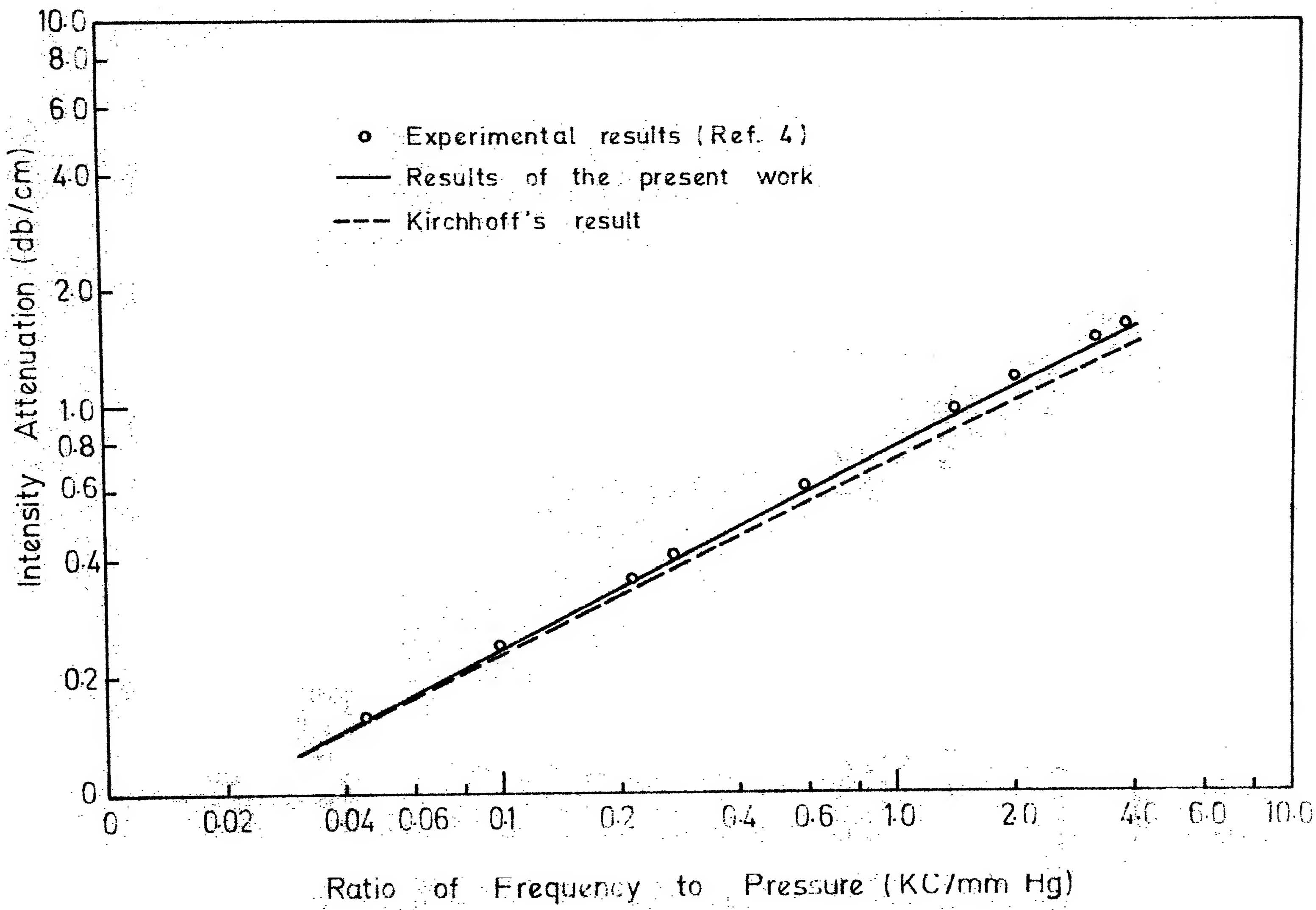

FIG. 2 SOUND ABSORPTION IN Ar (25.6°C)

V. REFERENCES

1. LORD RAYLEIGH, The Theory of Sound Vol. II, Dover Publications, New York, (1945).

2. P.S.H.HENRY, The Tube Effect in Sound Velocity Measurements, Proc. Phys. Soc. (London), 43, 340, (1931).

3. D.E. WESTON, The Theory of the Propagation of Plane Sound Waves in Tubes, Proc. Phys. Soc. (London), B-66, 695, (1953).

4. F. DOUGLAS SHIELDS, K.P.LEE, and W.J.WILEY, Numerical Solution for Sound Velocity and Absorption in Cylinderical Tubes, J. Acoust. Soc. Amer., 37, 724, (1965).

5. N. ROTT and M.M.OBERAI, An expausion Scheme for the Kirchhoff Theory Of Acoustic Damping in Long Cylinderical Tubes, (Private Communication).

ME-1972-M-RAT-EXT